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Holographic Integral Equations and Walking Technicolour

Raul Alvares, Nick Evans, Astrid Gebauer & George James Weatherill *

School of Physics and Astronomy
Southampton University
Southampton, SO17 1BJ
United Kingdom

Abstract

We study chiral symmetry breaking in the holographic D3-D7 system in a simple model with an arbitrary running coupling. We derive equations for the D7 embedding and show there is a light pion. In particular we present simple integral equations, involving just the running coupling and the quark self energy, for the quark condensate and the pion decay constant. We compare these to the Pagels-Stokar or constituent quark model equivalent. We discuss the implications for walking Technicolour theories. We also perform a similar analysis in the four dimensional field theory whose dual is the non-supersymmetric D3-D5 system and propose that it represents a walking theory in which the quark condensate has dimension $2 + \sqrt{3}$.

*rpa1e08@soton.ac.uk, evans@soton.ac.uk, ag806@soton.ac.uk, gjw@soton.ac.uk

1 Introduction

The D3-D7 system[1, 2, 3] in AdS-like spaces has allowed the study, through gauge/gravity duality or holography[4, 5, 6], of many aspects of strongly interacting gauge theories with quarks[7]. The system has been used to study quark confinement[8, 9], mesons[10], transport properties at finite temperature[11, 12, 13], and chiral symmetry breaking in the presence of a running coupling[14, 15, 16] or a magnetic field[17].

Here we wish to present a very simple model of chiral symmetry breaking and the associated Goldstone boson (essentially pion) in this system. The simple model consists of embedding the D7s in pure $AdS_5 \times S^5$ but with an arbitrary dilaton profile to represent the running coupling of the dual gauge theory. This basic model, although the metric is not back reacted to the dilaton's presence, provides a simple encapsulation of the chiral symmetry breaking mechanism in the D3-D7 system. In particular it will allow us to elucidate in the holographic equations of motion why there is a Goldstone boson present for the symmetry breaking. Further it will allow us to write integral equations for the parameters of the low energy chiral Lagrangian involving just the form of the running coupling and the quark self energy function (the D7 brane embedding function). These equations are very similar in spirit to the Pagels-Stokar formula[18] for the pion decay constant, f_π, and constituent quark model[19] estimates of the chiral condensate and so forth.

The formulae we will present for these low energy parameters allow one to develop intuition about how the low energy theory depends on the underlying gauge dynamics. We explore this and as a particular example look at walking[20, 21] Technicolour[22, 23] theories to see if the holographic model matches the folk lore from constituent quark models. Our results support the expectation that a walking regime will enhance the quark condensate relative to the pion decay constant.

In our final section we will perform a similar study for the non-supersymmetric D3/D5 system with a four dimensional overlap. We interpret this system as a walking gauge theory where the quark condensate has a dimension of $2+\sqrt{3}$ in the far UV. This theory is not of any obvious phenomenological use but the walking paradigm does seem to explain the physics of the system.

2 A simple D3-D7 chiral symmetry breaking model

We will consider a gauge theory with a holographic dual described by the Einstein frame geometry $AdS_5 \times S^5$

$$ds^2 = \frac{1}{g_{uv}} \left[\frac{r^2}{R^2} dx_4^2 + \frac{R^2}{r^2} \left(d\rho^2 + \rho^2 d\Omega_3^2 + dw_5^2 + dw_6^2 \right) \right] \tag{1}$$

where we have split the coordinates into the x_{3+1} of the gauge theory, the ρ and Ω_3 which will be on the D7 brane world-volume and two directions transverse to the D7, w_5, w_6. The radial coordinate, $r^2 = \rho^2 + w_5^2 + w_6^2$, corresponds to the energy scale of the gauge theory. The radius of curvature is given by $R^4 = 4\pi g_{uv}^2 N \alpha'^2$ with N the number of colours. g_{uv}^2 is the $r \to \infty$ value of the dilaton. In addition we will allow an arbitrary running as $r \to 0$ to represent the gauge theory coupling

$$e^{\phi} \quad = \quad g_{YM}^2(r^2) \quad = \quad g_{uv}^2 \; \beta(\rho^2 + w_5^2 + w_6^2) \tag{2}$$

where the function $\beta \to 1$ as $r \to \infty$. The $r \to \infty$ limit of this theory is dual to the $\mathcal{N} = 4$ super Yang-Mills theory and g_{uv}^2 is the constant large r asymptotic value of the gauge coupling.

We will introduce a single D7 brane probe[3] into the geometry to include quarks - by treating the D7 as a probe we are working in a quenched approximation although we can reintroduce some aspects of quark loops through the running coupling's form if we wish. Although this system only has a U(1) axial symmetry on the quarks corresponding to rotations in the $w_5 - w_6$ plane (formally this symmetry is an R-symmetry of the model but it is broken by a quark mass or condensate) we believe it is a good setting for studying the dynamics of the quark condensation. That process is driven by the strong dynamics rather than the global symmetries so the absence of a non-abelian axial symmetry should not be important[1].

We must find the D7 embedding function eg $w_5(\rho), w_6 = 0$. The Dirac Born Infeld action in Einstein frame is given by

$$\begin{aligned} S_{D7} &= -T_7 \int d^8\xi e^{\phi} \sqrt{-\det P[G]_{ab}} \\ &= -\overline{T_7} \int d^4x \; d\rho \; \rho^3 \beta \sqrt{1 + (\partial_\rho w_5)^2} \end{aligned} \tag{3}$$

where $T_7 = 1/(2\pi)^7 \alpha'^4$ and $\overline{T_7} = 2\pi^2 T_7 / g_{uv}^2$ when we have integrated over the 3-sphere on the D7. The equation of motion for the embedding function is therefore

$$\partial_\rho \left[\frac{\beta \rho^3 \partial_\rho w_5}{\sqrt{1 + (\partial_\rho w_5)^2}} \right] - 2 w_5 \rho^3 \sqrt{1 + (\partial_\rho w_5)^2} \frac{\partial \beta}{\partial r^2} = 0 \tag{4}$$

The UV asymptotic of this equation, provided the dilaton returns to a constant so the UV dual is the $\mathcal{N} = 4$ super Yang-Mills theory, has solutions of the form

$$w_5 = m + \frac{c}{\rho^2} + ... \tag{5}$$

where we can interpret m as the quark mass ($m_q = m/2\pi\alpha'$) and c is proportional to the quark condensate as we'll see below.

[1]The Sakai Sugimoto model[24] is an example of a gravity dual with a non-abelian chiral symmetry but it is fundamentally five dimensional and a clear prescription for including a quark mass is lacking - the result is that we would not know how to do this analysis in that model since we can not identify the quark self energy nor the quark condensate.

The embedding equation (4) clearly has regular solutions $w_5 = m$ when g_{YM}^2 is independent of r - the flat embeddings of the $\mathcal{N} = 2$ Karch-Katz theory[3]. Equally clearly if $\partial\beta/\partial r^2$ is none trivial in w_5 then the second term in (4) will not vanish for a flat embedding. We conclude that for any non-trivial gauge coupling the asymptotic solutions must contain the parameter c, a quark condensate. Whether $c \to 0$ or not as $m \to 0$ depends on the precise form of the running coupling chosen (note that $w_5 = 0$ is always a solution of (4)). However, if the coupling grows towards $r = 0$ as one would expect in a confining theory then there is clearly a growing penalty in the action for the D7 to approach the origin and we expect c to be non-zero.

As an example one can consider a gauge coupling running with a step of the form

$$\beta = a + 1 - a \tanh\left[\Gamma(r - \lambda)\right] \tag{6}$$

This form introduces conformal symmetry breaking at the scale $\Lambda = \lambda/2\pi\alpha'$ which triggers chiral symmetry breaking. The parameter a determines the increase in the coupling across the step but the solutions have only a small dependence on the value chosen because the area of increasing coupling is avoided by the D7 brane. An extreme choice of the profile is to let the coupling actually diverge at the barrier to represent the one loop blow up in the running of the QCD coupling - the solutions show the same behaviour as for a finite step provided the transition is not infinitely sharp. The parameter Γ spreads the increase in the coupling over a region in r of order Γ^{-1} in size - the effect of widening the step is to enhance the tail of the self energy function for the quark. We show the symmetry breaking embeddings in Figure 1.

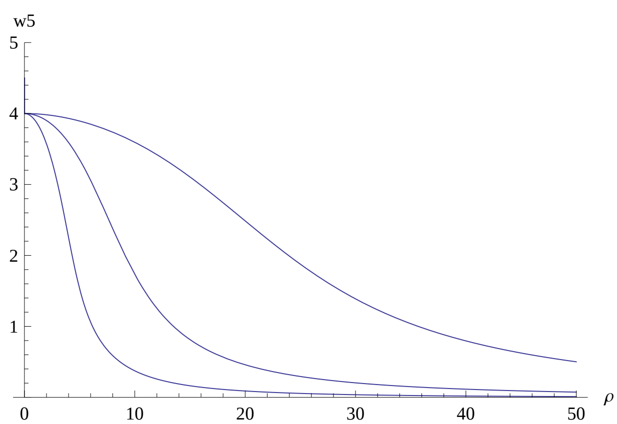

Figure 1: The D7 brane embeddings/quark self energy plots for the coupling ansatz in (6) - in each case the parameter $a = 3$ and from left to right: $\lambda = 3.19, \Gamma = 1$; $\lambda = 4.55, \Gamma = 0.3$; $\lambda = 10.4, \Gamma = 0.1$.

We will interpret the D7 embedding function as the dynamical self energy of the quark, similar to that emerging from a gap equation. The separation of the D7 from the ρ axis is the mass at some particular energy scale given by ρ - in the $\mathcal{N} = 2$ theory where the embedding is

flat the mass is not renormalized, whilst with the running coupling an IR mass forms - we have picked parameters in Figure 1 that generate the same dynamical quark mass at $\rho = 0$. We call the embedding function Σ_0 below.

2.1 Goldstone Mode

The embedding above lies at $w_6 = 0$ but there is clearly a set of equivalent solutions given by rotating that solution in the $w_5 - w_6$ plane. That degeneracy of the solutions is the vacuum manifold. We therefore expect a Goldstone mode associated with a fluctuation of the vacuum in the angular direction. For small fluctuations about the embedding above we may look at fluctuations in w_6. The quadratic order expanded action for such a small fluctuation is

$$S_7 \;=\; -\overline{T_7}\int \mathrm{d}\rho dx^4 \rho^3 \beta\sqrt{1+(\partial_\rho\Sigma_0)^2}\left(1+\frac{\partial_{r^2}\beta}{\beta}w_6^2+\frac{1}{2}\frac{(\partial_\rho w_6)^2}{1+(\partial_\rho\Sigma_0)^2}+\frac{1}{2}\frac{R^4}{r^4}(\partial_x w_6)^2+...\right) \quad (7)$$

note r, β and $\partial_{r^2}\beta$ are evaluated on the solution Σ_0 here and henceforth.

As usual we will seek solutions of the form $w_6(\rho, x) = f_n(\rho)e^{ik.x}$, $\; k^2 = -M_n^2$. Here n takes integer values - the solutions are associated with the Goldstone boson and its tower of radially excited states. The f_n satisfy the equation

$$\partial_\rho\left(\frac{\beta\rho^3\partial_\rho f_n}{\sqrt{1+(\partial_\rho\Sigma_0)^2}}\right) - 2\rho^3\sqrt{1+(\partial_\rho\Sigma_0)^2}(\partial_{r^2}\beta)f_n + \frac{1}{r^4}\rho^3\beta\sqrt{1+(\partial_\rho\Sigma_0)^2}R^4M_n^2 f_n = 0 \quad (8)$$

The presence of a Goldstone boson is now immediately apparent - there is a solution with $M_n^2 = 0$ and $f_0 = \Sigma_0$. With these substitutions the equation is exactly the embedding equation (4), a result of the symmetry between between w_5 and w_6. This is the pion like bound state of this theory - although there is only a broken U(1) axial symmetry, the absence of anomaly effects at large N make it closer in nature to the pions than the η' of QCD.

Naively the argument just given makes it appear there is a massless Goldstone for any w_5 solution including those where there is an explicit quark mass in the asymptotic fall off in (5). This is not the case though because to interpret the solution as a Goldstone requires f_0 to fall off at large ρ as $1/\rho^2$ - it must be a fluctuation in the condensate not the explicit mass. The naive massless solution is related to the fact that the theory has a spurious symmetry where $\bar{\psi}_L\psi_R \to e^{i\alpha}\bar{\psi}_L\psi_R$ and simultaneously $m \to e^{-i\alpha}m$. This spurious symmetry must be present in the string construction.

2.2 The low energy Chiral Lagrangian

The Goldstone field's low energy Lagrangian must take the form of a chiral Lagrangian, non-linear realization of the broken symmetry[25]. We can substitute the form $w_6 = f_0(\rho)\Pi(x) =$

$\Sigma_0\Pi(x)$ into (7) and integrate over ρ to obtain this Lagrangian

$$\begin{aligned}\mathcal{L} &= -\overline{T_7}\int \mathrm{d}\rho\rho^3\beta\sqrt{1+(\partial_\rho\Sigma_0)^2}\left(1+\frac{1}{2}\frac{R^4}{r^4}\Sigma_0^2(\partial_x\Pi)^2\right.\\ &\left.\qquad +\frac{1}{4}\frac{R^4}{r^4}\left[\frac{2}{\beta}\frac{d\beta}{dr^2}\Sigma_0^4+\frac{\Sigma_0^2(\partial_\rho\Sigma_0)^2}{1+(\partial_\rho\Sigma_0)^2}\right](\partial_x\Pi)^2\Pi^2+...\right)\end{aligned} \tag{9}$$

where we've used the equation of motion (8) to eliminate the second and third terms in (7) in the massless limit. We have also included the $\Pi^2(\partial_x\Pi)^2$ term from the fourth order expansion from which we will determine f_π.

This should be compared to the standard chiral Lagrangian form where $U=\exp(\sqrt{2}i\pi/f_\pi)$

$$\begin{aligned}\mathcal{L} &= V_0+\frac{f_\pi^2}{4}\partial_x U^\dagger\partial^x U+\mathcal{O}(p^4)\\ &= V_0+\frac{1}{2}(\partial_x\pi)^2+\frac{1}{f_\pi^2}(\partial_x\pi)^2\pi^2+\mathcal{O}(\pi^6)+\mathcal{O}(p^4)\end{aligned} \tag{10}$$

where V_0 is the vacuum energy and f_π is the pion decay constant.

We must rescale Π in (9) to the canonical normalization in (10) and then we can read off an integral expression for the pion decay constant. To ensure all factors of α' are absent from physical answers, as they must be, we must express our answer as the ratio of two physical scales. Here we will use the scale Λ in the gauge coupling running (6) that encodes the scale of the chiral symmetry breaking as our reference - we have

$$\frac{f_\pi^2}{\Lambda^2}=\frac{-N}{\pi^2\lambda^2}\left.\frac{\left[\int d\rho\rho^3\beta\sqrt{1+(\partial_\rho\Sigma_0)^2}\frac{\Sigma_0^2}{(\rho^2+\Sigma_0^2)^2}\right]^2}{\left[\int d\rho\rho^3\beta\sqrt{1+(\partial_\rho\Sigma_0)^2}\frac{1}{4(\rho^2+\Sigma_0^2)^2}\left(\frac{2}{\beta}\frac{d\beta}{dr^2}\Sigma_0^4+\frac{\Sigma_0^2(\partial_\rho\Sigma_0)^2}{1+(\partial_\rho\Sigma_0)^2}\right)\right]}\right|_{r^2=\rho^2+\Sigma_0^2} \tag{11}$$

Note that $\partial_{r^2}\beta$ is typically negative for the embeddings we have explored above so that f_π^2 is positive. Employing the embedding equation (4) the denominator may be simplified leaving

$$\frac{f_\pi^2}{\Lambda^2}=\frac{-4N}{\pi^2\lambda^2}\frac{\left[\int d\rho\rho^3\beta\sqrt{1+(\partial_\rho\Sigma_0)^2}\frac{\Sigma_0^2}{(\rho^2+\Sigma_0^2)^2}\right]^2}{\left[\int d\rho\frac{\Sigma_0^2}{(\rho^2+\Sigma_0^2)^2}\partial_\rho\left(\frac{\beta\rho^3\Sigma_0(\partial_\rho\Sigma_0)}{\sqrt{1+(\partial_\rho\Sigma_0)^2}}\right)\right]} \tag{12}$$

We can also extract an integral equation for the quark condensate (evaluated in the UV where there is no running) from our analysis. We use the fact that the expectation value of $\bar{q}_L q_R$ is given by $\frac{1}{Z}\frac{\partial Z}{\partial m_q}|_{m_q\to 0}$. For an infinitessimal value of m in the boundary embedding (5) we expect the full embedding, to leading order, to simply take the form $w_5=2\pi\alpha' m_q+\Sigma_0$. We insert this form into the vacuum energy and expand to leading order in m_q - the coefficient is just the quark condensate

$$\frac{\langle\bar{q}_L q_R\rangle}{\Lambda^3}=\frac{-N}{4\pi\lambda^3 g_{uv}^2 N}\left.\int d\rho\ \rho^3\Sigma_0\sqrt{1+(\partial_\rho\Sigma_0)^2}\partial_{r^2}\beta\right|_{r^2=\rho^2+\Sigma_0^2} \tag{13}$$

One may use the embedding equation (4) to turn this into a surface term that is then, given β becomes unity asymptotically, proportional to $\rho^3 \partial_\rho \Sigma_0|_{\rho\to\infty}$ which is just proportional to the constant c in (5) confirming the interpretation of c as the condensate. The integral form of the equation though allows intuition for the value of the condensate from the shape of the embedding as we will see. Note that if the 'tHooft coupling $g_{uv}^2 N$ is kept fixed both f_π and the condensate grow as N as expected.

The integral equations (12) and (13) that link low energy parameters to the underlying UV physics are our main results. They are very reminiscent of constituent quark model[19] results which input the quark self energy, $\Sigma(q)$, (for example determined from a gap equation[21]) to determine the same quantities. In particular those models give for the condensate

$$\langle \bar{q} q \rangle = \frac{N}{2} \int q^3 dq \frac{\Sigma}{q^2 + \Sigma^2} \tag{14}$$

and the Pagels Stokar formula[18] for the pion decay constant

$$f_\pi^2 = \frac{N}{8\pi^2} \int q^3 dq \frac{\Sigma^2 - \frac{1}{2} q^2 \Sigma \Sigma'}{(q^2 + \Sigma^2)^2} \tag{15}$$

where a prime indicates a derivatives with respect to q^2. Although our formulae are more complex and include the underlying gauge coupling's running there are nevertheless a number of common features. We will compare them for the case of walking Technicolour below.

It must be stressed that we have derived our expressions (12) and (13) in a toy holographic model of chiral symmetry breaking. Of course one can not just impose any random running of the gauge coupling and assume one is in a real gauge theory. We have also not included any back reaction of the space's metric to the presence of a non-trivial dilaton. The analysis is very similar in spirit to the chiral quark model assumption of an arbitrary choice of $\Sigma(q^2)$. Despite these flaws, we hope the simplicity of the expressions allows one to analytically understand the typical response of the holographic descriptions to different types of running coupling.

3 Walking Technicolour

The constituent quark model expressions (14) and (15) have underpinned much of the folk lore for walking Technicolour theories[20, 21]. In brief, in walking Technicolour the gauge coupling is assumed to transition from perturbative to non-perturbative behaviour at one scale, Λ_1 but then the running slows, only crossing some critical value for inducing chiral symmetry breaking at a scale, Λ_2, several orders of magnitude below Λ_1. In the period between Λ_1 and Λ_2 we imagine that the anomalous dimension ϵ of the quark condensate is negative (so $\bar{q}q$ has dimension less than three) - the condensate evaluated in the UV is then enhanced taking the rough value $\Lambda_2^{3-\epsilon} \Lambda_1^{\epsilon}$.

Gap equation analysis[21] provides an alternative but equivalent explanation for the enhancement of the quark condensate. There walking, which has a larger coupling value further into the UV, enhances the large q tail of the quark self energy $\Sigma(q)$. Looking at the constituent quark model expressions for low energy parameters one can see that f_π is dominated at small q (there is a q^4 in the denominator) and so f_π is broadly unchanged by walking. In a Technicolour model f_π sets the W and Z masses and hence the weak scale. On the other hand the condensate in (14) is given by a simple integral over $\Sigma(q)$ and hence grows if the tail of $\Sigma(q)$ is raised. The condensate is enlarged in walking theories relative to the weak scale. In extended Technicolour models[26] the condensate determines the standard model fermion masses - increasing it drives up the extended Technicolour scale, potentially suppressing flavour physics below current experimental bounds.

Do our holographic expressions agree with this story? The challenge is to simulate walking in a holographic setting. The problem is that we are always at strong coupling (large N) if we have a weakly coupled gravity dual. As we have seen, the introduction of any conformal symmetry breaking through the running coupling causes chiral symmetry breaking[2]. We can not therefore reproduce directly the physics at the scale Λ_1 discussed above where the theory moved to strong coupling but without causing chiral symmetry breaking.

As a first attempt to address this point we can be led by the solutions in Figure 1 as a result of the coupling ansatz in (6). If we increase the parameter Γ we effectively smear the scale at which the chiral symmetry breaking is induced over a range of $r \sim \Gamma^{-1}$. Could we use this smeared range to represent the separation between Λ_1 and Λ_2 above? The effect of the smearing is to enhance the tail of the self energy just as expected in walking theories.

If we now turn to the holographic expressions (12) & (13) we see that they naively share the same response to enhancing the tail of Σ_0 as the constituent quark model expressions (14) & (15) did to raising the tail of $\Sigma(q)$. In particular again f_π has a $1/\rho^4$ factor in the denominator of each integral involved, making it, one would expect, insensitive to changes in the tail of Σ_0. The expression for the condensate though is sensitive to the tail and should grow as walking is introduced. In fact though this analysis neglects the dependence of these functions on the derivatives of the gauge coupling and the self energy function Σ_0 - this additional understanding of dynamics coming from the gauge coupling running lies beyond the constituent quark model pictures. Both (12) and (13) are dominated around the points of maximum change in the coupling and Σ_0. Note though that the derivative of the coupling, $\partial_{r^2}\beta$, is evaluated on the brane, which in the cases above has precisely embedded itself so as to avoid large derivatives in β. By smoothing these functions through increasing Γ we include extra functional behaviour.

[2] Attempts to find backreacted holographic models of gauge theories with a walking profile such as those in [27, 28] could fall foul of this problem were they used to generate chiral symmetry breaking.

In fact these changes in the derivatives are more numerically important than the rise in the tail of Σ_0 for the plots in Figure 1. This means that the more "walking" looking self energies in fact give a slightly lower condensate for a fixed value of f_π. The simple coupling ansatz in (6) does not therefore accommodate a behaviour we can interpret in the usual walking picture. The model does suggest that there could be considerable variation in the ratio of the condensate to f_π in gauge theories with rather different speeds of IR running though. A recent lattice analysis suggest this ratio could vary as the number of quark flavours is changed in QCD [29].

To take advantage of the similarities between (12) & (13) and (14) & (15) one would need to keep the derivatives of the coupling and Σ_0 roughly fixed as the scale at which that change occurred was moved out to larger ρ. Our equations would in such a scenario provide the enhancement of the condensate that one looks for in a walking theory. Essentially one would want a self energy that rose sharply at large ρ but then flattened to meet the w_5 axis at the same value as the curves in Figure 1. This in fact matches the crucial signal of walking that one would expect $\Sigma_0(\rho = 0) \ll \Lambda$ with Λ the scale at which the high scale running occurs. Within holographic models this should be the crucial signal of walking.

This scenario suggests we are mimicking a slightly different walking dynamics in the gauge theory than that discussed above - imagine a theory in which the coupling ran to strong coupling (call this scale Λ_1 again) and then entered a conformal regime with coupling value slightly above the critical value needed to form a condensate. If the coupling was tuned from above sufficiently close to the critical value in its conformal window then a self energy would form but with a size considerably below Λ_1.

Realizing this sort of walking behaviour can be done in a straightforward, if adhoc, fashion. We need to break the symmetry between ρ and w_5, w_6 in the coupling ansatz β. A simple ansatz is just to shift our previous ansatz out to larger ρ:

$$\begin{aligned} \beta &= a + 1 - a \tanh\left[\Gamma(\sqrt{(\rho-\lambda_1)^2 + w_5^2 + w_6^2} - \lambda)\right] \qquad \rho \geq \lambda_1 \\ \beta &= 1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \rho < \lambda_1 \end{aligned} \tag{16}$$

This ansatz, which we sketch in Figure 2, leaves the derivative of β unchanged but shifted by λ_1 in ρ - this will ensure the condensate, which is given by (13) and dominated around λ_1 where the derivative of β is non-zero, will grow as λ_1^3. The embedding will still plateau around the same value of w_5 since above the step (which is quite sharp) the space is AdS and the embeddings must be flat. Below λ_1 the embedding becomes flat since the geometry is AdS (the first derivative of Σ_0 at $\rho = \lambda_1$ is smooth). Obviously this choice of β below λ_1 looks peculiar - one could though imagine that in that region there is a sharp step function to large coupling at small w_5, w_6 - the embeddings would remain the same.

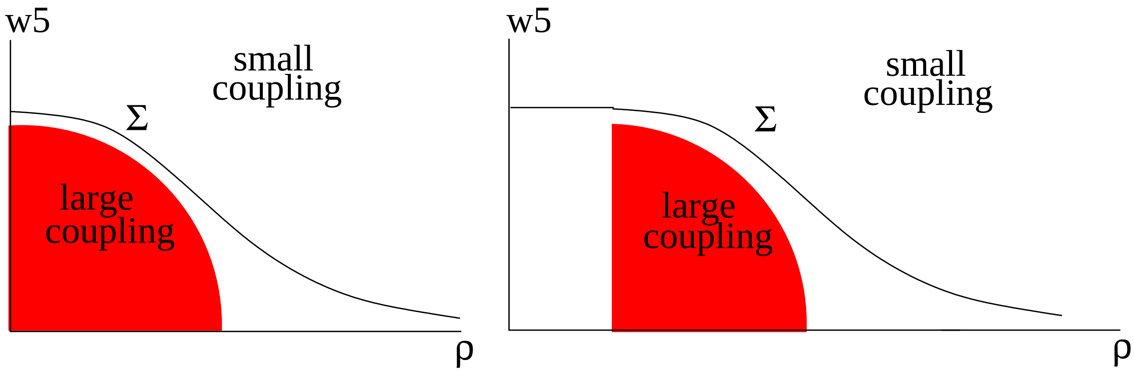

Figure 2: A sketch of the area in which the coupling is large in our ansatz in (16) and the resulting form of the embeddings Σ_0 - on the left for $\lambda_1 = 0$ and on the right for a non-zero λ_1

With the embeddings from this walking β ansatz we can analytically see how the expressions for f_π and $\langle \bar{q}q \rangle$ change with λ_1. In (12) the numerator will become independent of λ_1 as it grows whilst the denominator, which is proportional to the derivatives of Σ_0 and β will fall as $1/\lambda_1$. f_π will therefore scale as $\lambda_1^{1/2}$. The condensate expression (13) is dominated around λ_1 where the derivative of β is non-zero - it will grow as λ_1^3. Therefore if we raise λ_1 at fixed f_π the condensate will grow as $\lambda_1^{3/2}$. The rise is consistent with the usual claims that a walking theory will enhance the condensate.

It is also possible to numerically confirm this behaviour at least for small λ_1. In Figure 3 we show numerical embeddings, displaying the behaviour shown in Figure 2, as λ_1 is increased from 0 to 8. To keep the plateau value exactly equal we have tuned Γ in the coupling ansatz (it changes from 1 to 3.6 across these plots). The condensate grows by an order of magnitude across these plots and in the large λ_1 limit will presumably match the analytic behaviour discussed although more and more tuning of Γ would be needed.

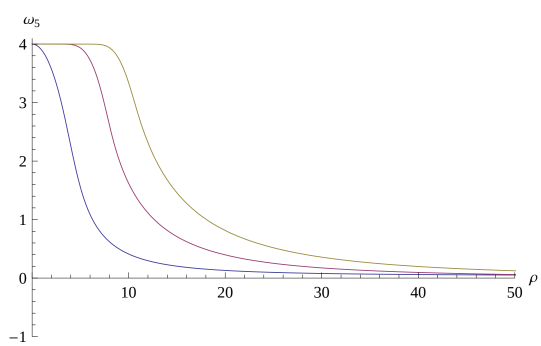

Figure 3: Numerically determined embeddings for the coupling ansatz in (16). These curves all have $a = 3$ and $\lambda = 3.19$ in addition the curves from from left to right correspond to the parameter choices $\lambda_1 = 0, \Gamma = 1$, $\lambda_1 = 5, \Gamma = 3.51$,$\lambda_1 = 8, \Gamma = 3.63$.

Note that breaking the symmetry between ρ and w_5, w_6 in the β ansatz is still consistent with the symmetries of the D3-D7 system. In fact interestingly a distinction between the ρ and w_5, w_6 directions is precisely what one would expect in a geometry backreacted to the D7 branes[1, 30]. It is therefore plausible that one could fine tune the number of quark flavours in some D3-D7 system to obtain these forms of ansatz for the dilaton.

4 The D3-D5 System

We now turn to an alternative attempt to describe aspects of walking dynamics with holography. On first meeting the D3/(probe)D7 system it seems as if that system should fundamentally be a walking gauge theory - the $\mathcal{N}=4$ gauge dynamics is conformal and strongly coupled in the UV. When we introduce running in the IR that triggers chiral symmetry breaking, should the physics not be closer in spirit to that of a walking theory rather than QCD? Why did we have to work so hard above to make that system walk? The reason it is not a walking theory is that the UV of the D3/D7 system possesses $\mathcal{N}=2$ supersymmetry which both forbids a quark condensate and protects the dimension of the $\bar{q}q$ condensate at three. That the self energy profiles Σ_0 fall off as $1/\rho^2$ in the analysis above is driven by that UV supersymmetry and mimics the behaviour of asymptotically free QCD.

It is natural then to look for a way to introduce quarks into $\mathcal{N}=4$ super Yang Mills which breaks supersymmetry even in the far UV. Using a D5 probe to introduce quarks seems the simplest example to explore. Here we consider the system with a four dimensional overlap of the D3 and the D5 not a three dimensional overlap as studied in [31]. Note that the strings between the D3 and D5 remain bi-fundamental fields of the gauge symmetry and global symmetry. The lowest energy modes of those strings are still at heart the gauge field, that would be present if the strings were free to move in the whole space, which become scalar fields, and the gaugino partners that become the fermionic quarks. In a non-supersymmetric theory the scalars will most likely become massive leaving fermionic quark multiplets in the $\mathcal{N}=4$ theory.

The metric of $AdS_5 \times S^5$ can be written in coordinates appropriate to the D5 embedding as:

$$ds^2 = \frac{1}{g_{uv}}\left[\frac{r^2}{R^2}\eta_{ij}dx^i dx^j + \frac{R^2}{r^2}\left(d\rho^2 + \rho^2 d\Omega_1^2 + d\omega_3^2 + d\omega_4^2 + d\omega_5^2 + d\omega_6^2\right)\right] \tag{17}$$

with $r^2 = \rho^2 + \omega_3^2 + \omega_4^2 + \omega_5^2 + \omega_6^2$ and $\rho^2 = \omega_1^2 + \omega_2^2$. R is the radius of AdS $R^4 = 4\pi g_{uv}^2 N\alpha'^2$ The D3 brane is extended in the x_i dimensions. The D5-brane will be extended in the ρ and Ω_1 directions. The $\omega_3, \omega_4, \omega_5$ and ω_6 are perpendicular to the D5-brane. g_{uv}^2 is the value of the dilaton for $r \to \infty$.

Let us first analyze the system with a constant dilaton

$$e^{\phi} = g_{uv}^2 \tag{18}$$

The action for a probe D5 brane assuming the embedding $\omega_5(\rho), \omega_3 = \omega_4 = \omega_6 = 0$ is:

$$\begin{aligned} S_{D5} &= -T_5 \int d^8\xi e^{\phi} \sqrt{-\det P\left[G\right]_{ab}} \\ &= -\overline{T_5} \int d^4x d\rho r^2 \rho \sqrt{1 + \left(\partial_\rho w_5\right)^2}, \end{aligned} \tag{19}$$

where $T_5 = 1/(2\pi)^5 \alpha'^3$ and $\overline{T_5} = T_5 2\pi/R^2 g_{uv}$. The embedding equation is

$$\partial_\rho \left[r^2 \rho \frac{(\partial_\rho \omega_5)}{\sqrt{1 + (\partial_\rho \omega_5)^2}} \right] - 2\omega_5 \rho \sqrt{1 + (\partial_\rho \omega_5)^2} = 0 \tag{20}$$

The large ρ behaviour of these solutions is

$$\omega_5 \sim m\rho^{\sqrt{3}-1} + c/\rho^{1+\sqrt{3}} \tag{21}$$

The full embeddings are shown on the left hand of Figure 4. Note that as $m \to 0$ in the UV asymptotics the full solutions lie along the ρ axis indicating that the condensate $c = 0$ and there is no spontaneous chiral symmetry breaking - this is a simple result of the absence of a scale in the conformal field theory.

We continue to interpret the parameter m in the D5 brane embedding as the quark mass. Then from equation (21) we can see that there is an effective anomalous dimension present for that mass - its dimension is $2 - \sqrt{3}$. The parameter c is then the quark condensate and has dimension $2+\sqrt{3}$. the change in the dimension of these operators in the UV conformal regime is exactly the physics that underlies the walking idea. Amusingly though here the anomalous dimension of the quark condensate is positive rather than negative as usually envisaged in walking theories. The D3/D5 system will not apparently be much use for constructing a phenomenological technicolour model. On the other hand here we are simply interested in testing the intuition for walking theories so we will continue to investigate for more formal reasons.

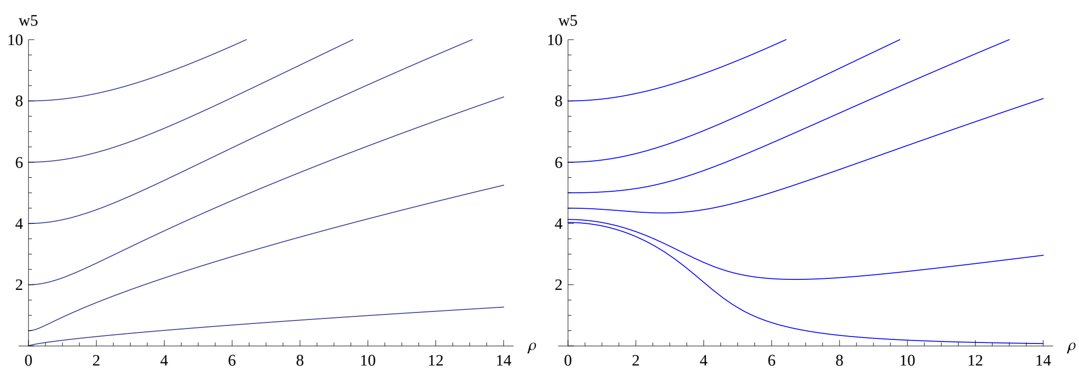

Figure 4: The regular embeddings of a D5 brane in pure AdS with $\beta = 1$ on the left. On the right the chiral symmetry breaking embeddings for the ansatz for β in (6) with $\Gamma = 1, \lambda = 3, a = 5$.

4.1 D3-D5 Embedding with a Non-Trivial Dilaton

Let us now include a non-trivial dilaton (gauge coupling) profile as we did above in the D3-D7 system

$$e^{\phi} = g^2_{YM}(r^2) = g^2_{uv}\beta(r^2). \tag{22}$$

For $r \to \infty$ $\beta \to 1$. The action is now

$$S_{D5} = -\overline{T_5} \int d^4x d\rho r^2 \beta \rho \sqrt{1 + (\partial_\rho w_5)^2}. \tag{23}$$

The embedding equation is

$$\partial_\rho \left[r^2 \beta \rho \frac{(\partial_\rho \omega_5)}{\sqrt{1 + (\partial_\rho \omega_5)^2}} \right] - 2\omega_5 \rho \sqrt{1 + (\partial_\rho \omega_5)^2} \left[\beta + r^2 (\partial_{r^2} \beta) \right] = 0 \tag{24}$$

The embeddings can be seen on the right in Figure 4 for the ansatz for β in (6). There is again chiral symmetry breaking with a non-zero $w_5(\rho = 0)$ as $m \to 0$ in the UV. The self energy curves fall off faster at large ρ which matches expectations from gap equations in a theory where the quark condensates dimension grows in the walking regime.

The embedding breaks the SO(4) symmetry in the $\omega_3 - \omega_6$ directions so we expect there to be Goldstone modes present. For example, there should be an equivalent solution when rotating the embedding in e.g. the $\omega_5 - \omega_6$ plane. Let's look at small fluctuations around the embedding Σ_0 in the ω_6 direction to find a Goldstone boson. The action for such fluctuations in quadratic order is

$$S_5 = -\overline{T_5} \int d^4x d\rho r^2 \beta \rho \sqrt{1 + (\partial_\rho \Sigma_0)^2} \left[1 + (\partial_{r^2}\beta) w_6^2 + \frac{1}{2} \frac{(\partial_\rho \omega_6)^2}{1 + (\partial_\rho \Sigma_0)^2} + \frac{1}{2} \frac{R^4}{r^4} (\partial_x \omega_6)^2 + \ldots \right] \tag{25}$$

where again r^2, β and $\partial_{r^2}\beta$ are all evaluated on the the D7 brane world volume Σ_0. We seek fluctuations of the form $\omega_6(x, \rho) = f_n(\rho) e^{ik\cdot x}$ with $k^2 = -M_n^2$. The equation of motion for the fluctuations give the following equations for f_n

$$\partial_\rho \left[r^2 \beta \rho \frac{(\partial_\rho f_n)}{\sqrt{1 + (\partial_\rho \Sigma_0)^2}} \right] + \frac{R^4}{r^2} \beta \rho \sqrt{1 + (\partial_\rho \Sigma_0)^2} M_n^2 f_n - 2\rho \sqrt{1 + (\partial_\rho \Sigma_0)^2} \left[\beta + r^2 \partial_{r^2} \beta \right] f_n = 0 \tag{26}$$

The equation with $M^2 = 0$ and $f_0 = \Sigma_0$ is the embedding equation (4) revealing the presence of the Goldstone mode.

The Lagrangian for the Goldstone field is found by writing $\omega_6 = f_0(\rho)\Pi(x) = \Sigma_0 \Pi(x)$ in (25) and integrating over ρ. We can expand $r^2\beta$ with $r^2 = \rho^2 + \Sigma_0^2 + \Sigma_0^2 \Pi^2$ as $r^2\beta(r^2) =$

$r^2\beta(r^2)|_{r^2=\rho^2+\Sigma_0^2} + \Sigma_0^2\Pi^2\partial_{r^2}(r^2\beta)|_{r^2=\rho^2+\Sigma_0^2}$ and then use the equation of motion (26) to eliminate the second and third terms in (25) for $M_n = 0$. This procedure gives the Lagrangian to quartic order

$$\mathcal{L} = -\overline{T_5}\int d\rho r^2\beta\rho\sqrt{1+(\partial_\rho\Sigma_0)^2}$$
$$\left[1+\frac{1}{2}\frac{R^4}{r^4}\Sigma_0^2(\partial_x\Pi)^2+\frac{1}{4}\frac{R^4}{r^4}\left(\frac{(\partial_\rho\Sigma_0)^2\Sigma_0^2}{1+(\partial_\rho\Sigma_0)^2}+2\Sigma_0^4\frac{\partial_{r^2}(\beta r^2)}{\beta r^2}\right)\left(\partial_x\Pi^2\right)\Pi^2+\dots\right]. \tag{27}$$

We can now rescale Π in (27) and get an expression for f_π. We find

$$\frac{f_\pi^2}{\Lambda^2} = \frac{-2N^{1/2}}{\pi^{3/2}\lambda^2}\frac{\left[\int d\rho\beta\rho\sqrt{1+(\partial_\rho\Sigma_0)^2}\frac{\Sigma_0^2}{\rho^2+\Sigma_0^2}\right]^2}{\left[\int d\rho\frac{\Sigma_0^2}{(\rho^2+\Sigma_0^2)^2}\partial_\rho\left(\frac{(\rho^2+\Sigma_0^2)\beta\rho\Sigma_0(\partial_\rho\Sigma_0)}{\sqrt{1+(\partial_\rho\Sigma_0)^2}}\right)\right]} \tag{28}$$

We also want to find out the value of the quark condensate. We expand $r^2\beta$ in (27) with $r = \rho^2 + (\Sigma_0 + m\rho^{\sqrt{3}-1})^2$ as $r^2\beta = r^2\beta|_{r^2=\rho^2+\Sigma_0^2} + \partial_{r^2}(r^2\beta)|_{r^2=\rho^2+\Sigma_0^2}(2m\rho^{\sqrt{3}-1}\Sigma_0 + \mathcal{O}(m^2))$. Then we can compare the vacuum energy, V_0, in (27) with the vacuum energy of the chiral Lagrangian to find the quark condensate

$$\frac{\langle\bar{q}q\rangle}{\Lambda^{2+\sqrt{3}}} = \frac{-N^{1/2}}{2g_{uv}^2N\pi^{1/2}\lambda^{2+\sqrt{3}}}\int d\rho\rho^{\sqrt{3}}\sqrt{1+(\partial_\rho\Sigma_0)^2}\Sigma_0\partial_{r^2}(r^2\beta)\bigg|_{r^2=\rho^2+\Sigma_0^2}. \tag{29}$$

These expressions for f_π and $\langle\bar{q}q\rangle$ are in some ways similar to those in the D3/D7 system. f_π is again dominated at low ρ whilst the condensate is more sensitive to the tail of Σ_0. In the D5 setting Σ_0 falls off more quickly in the UV and will suppress the condensate. This matches the chiral quark model results. On the other hand the factor of $N^{1/2}$ before each expression suggests some radical redistribution of the degrees of freedom in the UV conformal regime which we can offer no explanation for.

It is important to also note that one can not directly compare the condensates in the D5 and D7 cases since they have different intrinsic dimension even in the far UV. In fact to convert the D3/D5 theory to the usual walking set up would require the inclusion of extra UV physics (equivalent to that at the scale Λ_1 in the walking discussion above) where the condensate's dimension changes to three. The condensate above that scale would be suppressed by a further factor of roughly $\Lambda_1^{\sqrt{3}-1}$.

Whilst the D3/D5 system may not form the basis of any helpful phenomenological model we do believe that the walking paradigm is the correct way to interpret the system and the anomalous dimensions present in the UV.

5 Conclusions

We have presented a general description of chiral symmetry breaking in the D3/D7 system that describes a strongly coupled gauge theory with quarks. The model allows one to compute the dependence of the parameters of the low energy chiral Lagrangian on the running coupling or dilaton form. Our integral formulae for f_π and the quark condensate allow analytic understanding of how these quantities depend on the coupling and the dynamical mass of the quark in a similar way to the results of chiral quark models and the Pagels-Stokar formula. Our model is not complete since we do not back react the geometry to the dilaton. However, we view this as a necessary evil to construct intuition in this type of set up to the response to different dilaton profiles. This toy environment should provide good guidance for those wishing to construct fully backreacted solutions that show specific phenomena.

We have used our results to understand how walking like gauge dynamics could be included in a holographic framework. The crucial signal of walking should be that the quark self energy at zero momentum should be much less than the scale at which conformal symmetry breaking is introduced. We displayed in figure 2 the form a dilaton profile must take to achieve walking. Our integral equations support the usual hypothesis that walking in a gauge theory would tend to boost the value of the quark condensate relative to the value of f_π.

Finally we studied the non-supersymmetric D3/D5 system with a four dimensional overlap and proposed that the conformal UV of the theory should be considered as a walking phase of a gauge theory. The anomalous dimensions of the quark mass and condensate were computed - in this theory the dimension of the quark condensate is $2+\sqrt{3}$ which is greater than the canonical dimension 3. Normally walking is constructed to lower this dimension but this theory hopefully nevertheless adds to our knowledge of walking behaviour.

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References

[1] M. Grana and J. Polchinski, Phys. Rev. **D65** (2002) 126005, [arXiv: hep-th/0106014].

[2] M. Bertolini, P. Di Vecchia, M. Frau, A. Lerda and R. Marotta, Nucl. Phys. B **621**, 157 (2002) [arXiv:hep-th/0107057].

[3] A. Karch and E. Katz, JHEP **0206**, 043 (2002) [arXiv:hep-th/0205236].

[4] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998) Int. J. Theor. Phys. **38**, 1113 (1999) [arXiv:hep-th/9711200].

[5] E. Witten, Adv. Theor. Math. Phys. **2** (1998) 253 [arXiv:hep-th/9802150].

[6] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Phys. Lett. B **428** (1998) 105 [arXiv:hep-th/9802109].

[7] J. Erdmenger, N. Evans, I. Kirsch and E. Threlfall, Eur. Phys. J. A **35** (2008) 81 [arXiv:0711.4467 [hep-th]].

[8] J. M. Maldacena, Phys. Rev. Lett. **80** (1998) 4859 [arXiv:hep-th/9803002].

[9] S. J. Rey and J. T. Yee, Eur. Phys. J. C **22** (2001) 379 [arXiv:hep-th/9803001].

[10] M. Kruczenski, D. Mateos, R. C. Myers and D. J. Winters, JHEP **0307** 049, 2003 [arXiv:hep-th/0304032].

[11] D. T. Son and A. O. Starinets, Ann. Rev. Nucl. Part. Sci. **57** (2007) 95 [arXiv:0704.0240 [hep-th]].

[12] S. S. Gubser and A. Karch, arXiv:0901.0935 [hep-th].

[13] D. Mateos, R. C. Myers and R. M. Thomson, JHEP **0705** (2007) 067 [arXiv:hep-th/0701132].

[14] J. Babington, J. Erdmenger, N. J. Evans, Z. Guralnik and I. Kirsch, Phys. Rev. D **69** (2004) 066007 [arXiv:hep-th/0306018].

[15] K. Ghoroku and M. Yahiro, Phys. Lett. B **604** (2004) 235 [arXiv:hep-th/0408040].

[16] M. Kruczenski, D. Mateos, R. C. Myers and D. J. Winters, JHEP **0405** (2004) 041 [arXiv:hep-th/0311270].

[17] V. G. Filev, C. V. Johnson, R. C. Rashkov and K. S. Viswanathan, JHEP **0710** (2007) 019 [arXiv:hep-th/0701001].

[18] H. Pagels and S. Stokar, Phys. Rev. D **20** (1979) 2947.

[19] B. Holdom, J. Terning and K. Verbeek, Phys. Lett. B **245** (1990) 612.

[20] B. Holdom, Phys. Rev. D **24**, 1441 (1981).

[21] T. W. Appelquist, D. Karabali and L. C. R. Wijewardhana, Phys. Rev. Lett. **57** (1986) 957.

[22] S. Weinberg, Phys. Rev. D **13** (1976) 974.

[23] L. Susskind, Phys. Rev. D **20** (1979) 2619.

[24] T. Sakai and S. Sugimoto, Prog. Theor. Phys. **113** (2005) 843 [arXiv:hep-th/0412141].

[25] S. R. Coleman, J. Wess and B. Zumino, Phys. Rev. **177** (1969) 2239.

[26] E. Eichten and K. D. Lane, Phys. Lett. B **90** (1980) 125.

[27] C. Nunez, I. Papadimitriou and M. Piai, arXiv:0812.3655 [hep-th].

[28] D. Elander, C. Nunez and M. Piai, arXiv:0908.2808 [hep-th].

[29] T. Appelquist *et al.*, arXiv:0910.2224 [Unknown].

[30] I. Kirsch and D. Vaman, Phys. Rev. D **72** (2005) 026007 [arXiv:hep-th/0505164].

[31] A. Karch and L. Randall, JHEP **0106** (2001) 063 [arXiv:hep-th/0105132].